Lecture 2: Equity Premium

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Overview

- Some basic facts
- Study the asset pricing implications of household portfolio choice
- Consider the quantitative implications of a second-order approximation to asset return equations
- Reference: Mehra and Prescott (JME, 1984)

The Puzzle

UNITED STATES ANNUAL DATA, 1889-1978

- average real consumption growth rate 1.8% per year
- real consumption growth rate standard deviation 2%
- average real equity return 8% per year
- average real riskless rate 1% per year (US Tbill - Inflation)
- equity premium in the data is large 7% per year on an annual basis
- Question: Can a simple model generate this equity premium?

Mehra and Prescott Economy

- Two assets only (simplification of Lucas economy) and a representative household
- One asset is risk free and pays return R_t^f known at time t
- The other asset is risky and pays return R_t, unknown at time t realized at time $t+1$.
- Representative household makes portfolio choices, chooses to invest quantity B_t in the riskless asset, and quantity S_t in the risky asset, to maximize

$$E_t \sum_{s=0}^{\infty} \beta^s u(c_{t+s}), \; 0 < \beta < 1$$

- subject to the budget constraint

$$B_{t+1} + S_{t+1} + c_t = R_t^f B_t + R_t S_t$$

Optimal portfolio choices

$$B_{t+1} : u'(c_t) = R^f_{t+1}\beta E_t\left[u'(c_{t+1})\right]$$

$$S_{t+1} : u'(c_t) = \beta E_t\left[u'(c_{t+1})R_{t+1}\right]$$

- Rearranging we have:

$$1 = R^f_{t+1}E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]$$

$$1 = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}R_{t+1}\right]$$

Risk Neutrality

- $u'(c_t) = \text{constant}$
- The pricing equations become

$$R^f_{t+1} = E_t R_{t+1}$$

General case

- Pricing equation implies

$$E_t\left[m_{t+1}R_{t+1}\right] = 1$$

where m_{t+1} is discount factor or pricing kernel and R_{t+1} is the return.

- The pricing equation implies the discount factor depends on consumption:

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

Implications (1)

- For stocks

$$P_t = E_t \left\{ m_{t+1} X_{t+1} \right\}$$

where $X_{t+1} = (D_{t+1} + P_{t+1})$ and $R_{t+1} = \frac{D_{t+1}+P_{t+1}}{P_t}$

- Recursive substitution

$$P_t = E_t \left\{ m_{t+1} \left(D_{t+1} + E_{t+1} \left\{ m_{t+2} \left(D_{t+2} + P_{t+2} \right) \right\} \right) \right\}$$

- With risk neutrality $m_{t+s} = \beta^s$ and using the law iterated expectations

$$P_t = E_t \left\{ \sum_{s=0}^{\infty} \beta^s D_{t+s} \right\}$$

- **observation**: the law iterated expectations says that $(E_t z_{t+2} = E_t E_{t+1} z_{t+2})$

Implications (2)

- Let $\beta = 1/(1+r)$ and assume D_t is random walk with drift

$$D_t = (1+g)\,D_{t-1} + \varepsilon_t,$$

  where ε_t is white noise with mean 0 and variance σ_ε.
- Then

$$E_t D_{t+s} = (1+g)^s\,D_t,\ \text{all}\ t$$

- This implies

$$P_t = \sum_{s=0}^{\infty} \left(\frac{1+g}{1+r}\right)^s D_t$$

- Assume $\frac{1+g}{1+r} < 1$. Then:

$$\left(1 - \frac{1+g}{1+r}\right) P_t = D_t$$

or

$$\frac{P_t}{D_t} = \frac{1+r}{r-g}$$

Implications (3)

- For risk-free one-period bond that pays one unit of consumption tomorrow:

$$P_t = E_t \{m_{t+1}\}$$

where

$$R_{t+1} = \frac{1}{P_t}$$

- Nominal securities:

$$1 = E_t \left\{ m_{t+1} \frac{X_{t+1}^n}{P_t^n} \frac{1}{1 + \pi_{t+1}} \right\}$$

where

$$R_{t+1}^n = \frac{X_{t+1}^n}{P_t^n}$$

is the nominal return and

$$1 + \pi_{t+1} = \frac{P_{t+1}^{CPI}}{P_t^{CPI}}$$

Consumption-Based Asset Pricing (1)

- Using pricing equation for risky and riskless assets:

$$E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}R_{t+1}\right] = R^f_{t+1}E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]$$

using the covariance formula

$$cov(x,y) = E\left[(x-Ex)(y-Ey)\right] = E(xy) - ExEy$$

$$cov_t(m_{t+1}, R_{t+1}) + E_t m_{t+1} E_t R_{t+1} = R^f_{t+1} E_t m_{t+1}$$

and rearranging gives

$$\left(E_t R_{t+1} - R^f_{t+1}\right) E_t m_{t+1} = -cov_t(m_{t+1}, R_{t+1})$$

Consumption-Based Asset Pricing (2)

- Using pricing equation for riskless asset

$$E_t m_{t+1} = \frac{1}{R^f_{t+1}}$$

in the equation

$$\left(E_t R_{t+1} - R^f_{t+1}\right) E_t m_{t+1} = -cov_t\left(m_{t+1}, R_{t+1}\right)$$

get

$$\frac{E_t R_{t+1} - R^f_{t+1}}{R^f_{t+1}} = -cov_t\left(m_{t+1}, R_{t+1}\right)$$

Consumption-Based Asset Pricing (3)

- If the risky return covaries positively with tomorrow's consumption, c_{t+1}, then the LHS is positive and the asset return bears a positive premium over the risk free rate.
- If the risky return covaries negatively with tomorrow's consumption then the LHS is negative and the asset return bears a negative premium over the risk free rate.
- Intuition: assets whose returns have a negative covariance with consumption provide a hedge against consumption risk. Households are willing to accept a lower expected return since these assets provide insurance against low future consumption.

The equity premium puzzle

- Assume

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

- The Euler equations are

$$c_t^{-\gamma} = R_{t+1}^f E_t \beta c_{t+1}^{-\gamma}$$

$$c_t^{-\gamma} = E_t \left[R_{t+1} \beta c_{t+1}^{-\gamma} \right]$$

Approximation to the Euler equation

- Let $x_{t+1} = \ln(c_{t+1}) - \ln(c_t)$, $r_{t+1} = \ln(R_{t+1})$, the Euler equations become:

$$1 = \beta E_t \exp\left(-\gamma x_{t+1} + r^f_{t+1}\right)$$

$$1 = \beta E_t \left(-\gamma x_{t+1} + r_{t+1}\right)$$

- Assume that consumption growth and asset returns are jointly normally distributed:

$$\begin{bmatrix} x_{t+1} \\ r_{t+1} \end{bmatrix} \sim N\left(\begin{bmatrix} \overline{x}_{t+1} \\ \overline{r}_{t+1} \end{bmatrix}, \begin{bmatrix} var\left(x_{t+1}\right) & cov\left(x_{t+1}, r_{t+1}\right) \\ cov\left(x_{t+1}, r_{t+1}\right) & var\left(r_{t+1}\right) \end{bmatrix}\right)$$

Remember that if $x \sim N\left(\overline{x}, \sigma_x^2\right)$ then $X = \exp(x)$ is log-normal distributed with

$$E\left(X\right) = \exp(\overline{x} + \frac{1}{2}\sigma_x^2)$$

Risk premium with log-normal distribution

- The Euler equations become

$$1 = \beta \exp\left(-\gamma \overline{x}_{t+1} + r^f_{t+1} + \frac{1}{2} var(-\gamma x_{t+1})\right)$$

$$1 = \beta \exp\left(-\gamma \overline{x}_{t+1} + \overline{r}_{t+1} + \frac{1}{2} var(-\gamma x_{t+1} + r_{t+1})\right)$$

Implications (1)

- Take logs and equate these equations:

$$-\gamma \bar{x}_{t+1} + r^f_{t+1} + \frac{1}{2} var(-\gamma x_{t+1}) = -\gamma \bar{x}_{t+1} + \bar{r}_{t+1} + \frac{1}{2} var(-\gamma x_{t+1} + r_{t+1})$$

$$\begin{aligned} \bar{r}_{t+1} - r^f_{t+1} &= \frac{1}{2}\left[var(-\gamma x_{t+1}) - var(-\gamma x_{t+1} + r_{t+1})\right] \\ &= -\frac{1}{2} var(r_{t+1}) + \gamma cov(x_{t+1}, r_{t+1}) \end{aligned}$$

where we used the formula

$$var(y+z) = var(y) + var(z) + 2cov(y,z)$$

Implications (2)

- As $R_{t+1} = \log r_{t+1}$ then $\log E_t R_{t+1} = \overline{r}_{t+1} + \frac{1}{2} var(r_{t+1})$ and replace in

$$\overline{r}_{t+1} - r^f_{t+1} = -\frac{1}{2} var(r_{t+1}) + \gamma cov_t(x_{t+1}, r_{t+1})$$

to get

$$\log E_t R_{t+1} - \log R^f_{t+1} = \gamma corr_t(x_{t+1}, r_{t+1}) \sigma_{x_{t+1}} \sigma_{r_{t+1}}$$

where we used the formula
$corr_t(x_{t+1}, r_{t+1}) = cov_t(x_{t+1}, r_{t+1}) / (\sigma_{x_{t+1}} \sigma_{r_{t+1}})$

Quantitative implications

- The equity premium is:

$$\log E_t R_{t+1} - \log R^f_{t+1} = \gamma corr(x_{t+1}, r_{t+1}) \sigma_{x_{t+1}} \sigma_{r_{t+1}}$$

- In US data, $\sigma_r = 0.167$, $\sigma_x = 0.036$, $corr(x, r) = 0.4$ so
  - If $\gamma = 1$ we have $\log E_t R_{t+1} - \log R^f_{t+1} = 0.24\%$.
  - If $\gamma = 10$ we have $\log E_t R_{t+1} - \log R^f_{t+1} = 2.4\%$.
  - If $\gamma = 25$ we have $\log E_t R_{t+1} - \log R^f_{t+1} = 6.0\%$.

too high? (1)

Example 1. What would be the interest rate that would make a household that earns $50,000$ euros per year willing to postpone an annual vacation that costs $3,000$ euros?

$$R^f_{t+1} = \beta E_t \left(\frac{c_{t+1}}{c_t} \right)^{\gamma}$$

taking $\beta = 1, \gamma = 25$ and no uncertainty in the income process get

$$R^f_{t+1} = \left[\frac{\beta u'(c_{t+1})}{u'(c_t)} \right]^{-1} = \left(\frac{53,000}{47,000} \right)^{25} = 20.16 \approx 2016\%$$

too high? (2)

Example 2. The Certainty Equivalent, *CE*, of a lottery that gives 50, 000 euros with 50% probability or 100, 000 euros with 50% probability

$$\frac{(CE)^{1-\gamma}}{1-\gamma} = \frac{1}{2}\frac{(50,000)^{1-\gamma}}{1-\gamma} + \frac{1}{2}\frac{(100,000)^{1-\gamma}}{1-\gamma}$$

$$\begin{aligned}
\gamma &= 0 \quad & CE &= 75,000 \\
\gamma &= 1 \quad & CE &= 70,711 \\
\gamma &= 2 \quad & CE &= 66,246 \\
\gamma &= 5 \quad & CE &= 58,566 \\
\gamma &= 10 \quad & CE &= 53,991 \\
\gamma &= 20 \quad & CE &= 51,858 \\
\gamma &= 30 \quad & CE &= 51,209
\end{aligned}$$

Quantitative implications: Risk free rate

- From Euler equation

$$1 = \beta \exp\left(-\gamma \bar{x}_{t+1} + r^f_{t+1} + \frac{1}{2} var(-\gamma x_{t+1})\right)$$

  get the risk free rate is:

$$r^f_{t+1} = -\log\beta + \gamma \bar{x}_{t+1} - \frac{\gamma^2}{2} var(x_{t+1})$$

- Suppose $\beta = 0.999$ (in order for r^f_{t+1} to be as small as possible), $x = 0.015$ (tx. cresc. cons.), $\sigma_x = 0.036$ then we need $\gamma = 0.6$ to get $r^f_{t+1} = 1\%$.
  - If $\gamma = 10$ get $r^f_{t+1} = 22\%$
  - If $\gamma = 25$ get $r^f_{t+1} = 78\%$
- This is opposite to equity-premium puzzle – we need very low γ to match risk-free rate

Additional quantitative implications: Risk free rate

- If consumption growth is iid and homoskedastic, then risk free rate is constant.
- Risk free rate is higher if consumers are more impatient (have a high relative preference for consumption in the present) i.e. have a low β (time preference)
- Risk free rate is higher when expected consumption growth ($\overline{x}_{t+1}$) is higher (intertemporal substitution).
- Risk free rate is low when conditional consumption volatility ($var(c_{t+1})$) is high (precautionary savings).

Conclusions:

1. The simple model cannot explain the level of the equity premium.
2. It was first formalized by Mehra and Prescott in 1984. It remains a mystery: the difference is too large to reflect a "proper" level of compensation that would occur as a result of investor risk aversion
3. Possible explanations:

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    - (i) A large differential in the cost of trading between the stock and bond markets;
    - (ii) more general preferences that allow for the separation between risk aversion and IMRS;
    - (iii) incomplete markets (representative agent),
    - (iv) borrowing constraints,
    - (v) market segmentation (heterogeneity of agents).

Power utility function

Observation:
With power utility function the γ controls both:
1. The **intertemporal marginal rate of substitution** (changes of consumption across time due to changes in the relative price of consumption today vs future)
2. **Risk aversion** (changes of consumption across states of nature, which applies only when there is uncertainty).
3. **Precautionary savings** has to do with the volatility of consumption, and is related with the 3rd derivative of the utility function

More facts on the equity premium (1)

- **Fact 1**: Risk premiums also vary over time, with a clear business-cycle correlation. Stock, bond, and currency returns can be forecasted by regressions of the form

$$R^e_{t+1} = a + by_t + \varepsilon_{t+1}$$

  using as the forecasting variable y_t the price/dividend or price/earnings ratio of stocks, yield spreads of bonds, or interest rate spreads across countries.

- Expected returns vary over time as much as their level: $\sigma(E_t(R^e_{t+1})) = \sigma\left(a + by_t\right)$ is large compared to $E(R^e)$: If the equity premium is 6% on average, it is as likely to be 1% or 11% at any moment in time. (A regression of returns on dividend yields gives a standard error of expected returns $\sigma(E_t(R^e_{t+1})) = \sigma\left(a + by_t\right)$ of 5.5 percentage points.)

More facts on the equity premium (2)

- **Fact 2:** Furthermore, expected returns are high, prices are low, and risk premiums are high, in a coordinated way across many asset classes, in the bottoms of recessions. Conversely,expected returns are low, prices are high, and risk premiums are low at the tops of booms.
- **Problem:** need a model that can explain simultaneously a high equity premium and a higher equity premium in bad times.

How to fix the model?

To explain these facts, the macro-finance literature explored a wide range of alternative preferences and market structures.

1. Habits (Campbell and Cochrane, 1999).
2. Recursive utility (Epstein and Zin, 1989).
3. Long run risks (Bansal and Yaron, 2004; Bansal, Kiku, and Yaron, 2012).
4. Idiosyncratic risk (Constantinides and Duffie, 1996).
5. Heterogeneous preferences (Gârleanu and Panageas, 2015).
6. Rare Disasters (Reitz, 1988; Barro, 2006).
7. Utility nonseparable across goods (Piazzesi, Schneider, and Tuzel, 2007).
8. Leverage; balance-sheet; (Brunnermeier, 2009; Krishnamurthy and He, 2013).
9. Ambiguity aversion, min-max preferences, (Hansen and Sargent, 2011).
10. Behavioral finance; probability mistakes (Shiller, 1981, 2014).

Intuition (1)

- These approaches look different, but in the end the ideas are quite similar. The strategies in the literature boil down to generalize the discount factor to

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} Y_{t+1}$$

  where the new variable Y_{t+1} does most of the work, or allow the probability of future states to depend on the current state in a particular way - higher probability of a future disaster if the current state is bad.

Intuition (2)

- In the habit model, endogenous time-varying individual risk aversion is at work - people are less willing to take risks in bad times.
- In long-run risks, rare disasters and idiosyncratic risks models, the risk itself is time-varying.
- In heterogeneous agent models, the market has a time-varying risk-bearing capacity, though neither risks, or individual risk aversion, or individual probability mis-perceptions need vary over time.
- In heterogeneous agent models, changes in the wealth distribution that favor more or less risk averse agents induces the shift in risk-bearing capacity.